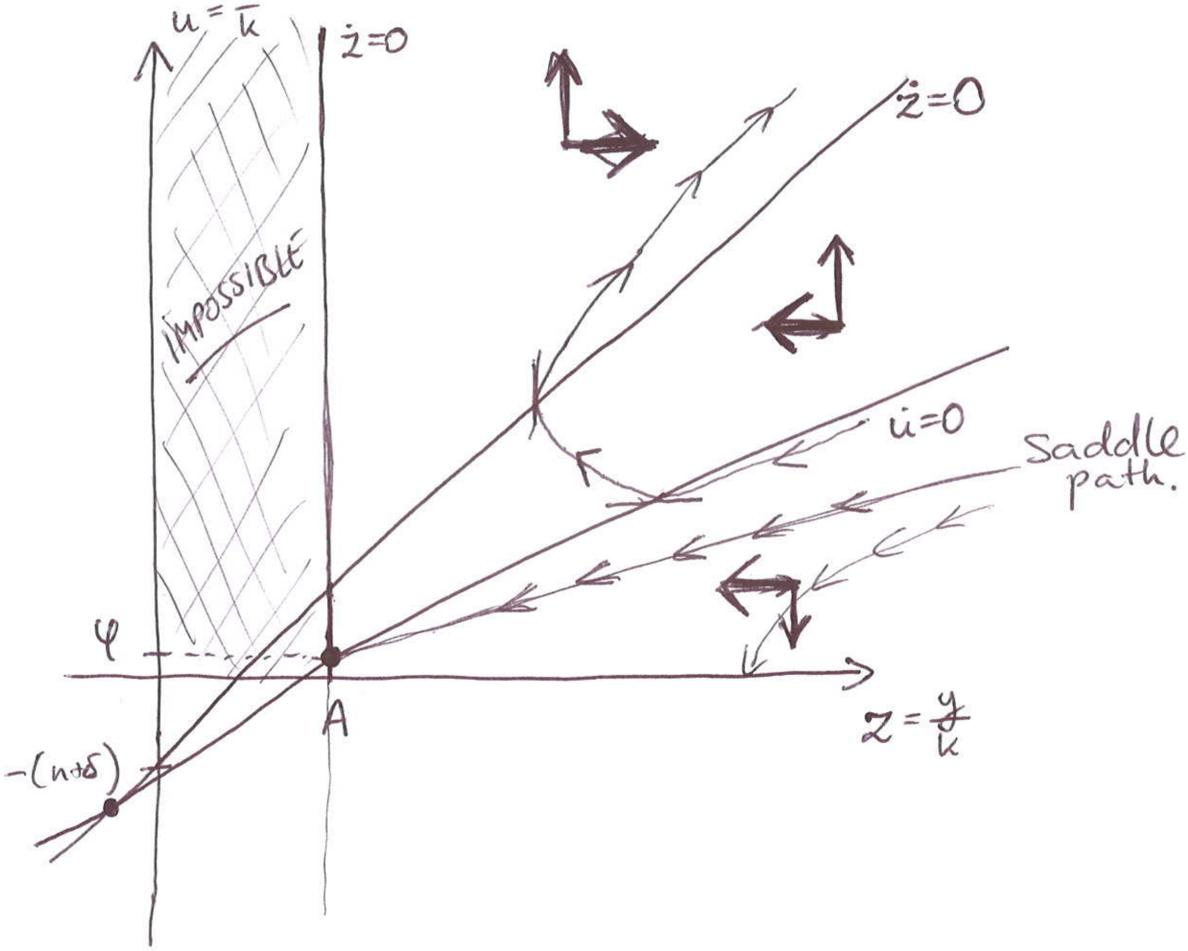
Isoclines ($\dot{u}=0$ and $\dot{z}=0$)

(1) $\dot{u}=0 \Leftrightarrow \hat{u}=0 \Leftrightarrow u = z\left(1-\frac{\alpha}{\theta}\right) - \frac{1-\alpha}{\theta}A + \frac{\delta+\rho}{\theta} - n - \delta$

(2) $\dot{z}=0 \Leftrightarrow \hat{z}=0 \Leftrightarrow z=A$ or $u = z-(n+\delta)$



Transversality condition:

$$\lim_{t\to\infty} \lambda k = \lim_{t\to\infty} k(t)e^{-\int_0^t (A+Bk(v)^{\alpha-1}-\delta-n)\,dv}.$$

- If the system converges to the steady state, then

$$\widehat{\lambda k} = \hat{\lambda} + \hat{k} = -(r-n) + \frac{r-\rho}{\theta} = r\left(\frac{1}{\theta}-1\right) - \frac{\rho}{\theta} + n =$$

$$= (A-\delta)\left(\frac{1}{\theta}-1\right) + \underbrace{Bk^{\alpha-1}\left(\frac{1}{\theta}-1\right)}_{\to 0 \text{ as } k\to\infty} - \frac{\rho}{\theta} + n = -\varphi \underset{\text{BY ASSUMPTION.}}{<} 0$$

Negative growth rate implies that the formula tends to 0. □

- Otherwise, the TVC is violated.

- Note that the capital's share of output, $\pi_k = \frac{\tilde{r}k}{Y} = \frac{Ak + B\alpha k^{\alpha}}{Ak + Bk^{\alpha}}$

and so $\pi_k = \frac{A + B\alpha k^{\alpha-1}}{A + Bk^{\alpha-1}}$ $\xrightarrow{k\to\infty} 1$, $\xrightarrow{k\to 0_+} \alpha$,

π_k gradually increases from α (the neoclassical case) to 1 (the AK case).

A simplified version of the Uzawa-Lucas model.

(JUST TO SHOW THE MECHANISM OF LONG-RUN GROWTH DRIVEN BY HUMAN CAPITAL ACCUMULATION)

!!

$$\begin{cases} Y = A K^{\alpha} H_Y^{1-\alpha} \\ \dot{K} = Y - C - \delta K = sY - \delta K & [s \text{ — should be endogenous}] \\ \dot{H} = \gamma H_H - \delta_H H & [H_H \text{ — — || — }] \\ H = H_H + H_Y & (H_H \text{ - teachers, } H_Y \text{ - other workers}) \end{cases}$$

Let us focus on the balanced growth path (BGP).

- **Assumption**: at the BGP, all variables grow at a fixed rate.

(1) $\hat{H} = \frac{\dot{H}}{H} = \gamma\left(\frac{H_H}{H}\right) - \delta_H := \gamma u - \delta_H$ (u – share of teachers in total employment)

We "hope" that in equilibrium, $\hat{H} = \gamma u^* - \delta_H > 0$.
Then, H accumulation will be the ultimate source of growth.

(2) $\hat{Y} = \underset{0}{\hat{A}} + \alpha \hat{K} + (1-\alpha)\hat{H}_Y = \alpha \hat{K} + (1-\alpha)\hat{H} + (1-\alpha)\widehat{(1-u)}$

Note: $H_Y = (1-u)H$
$H_H = uH$

(3) Assume $s \equiv const$, $u \equiv const$, $A \equiv const$.

(4) $\hat{K} = s\frac{Y}{K} - \delta \Rightarrow \frac{Y}{K} \equiv const \Rightarrow \hat{Y} = \hat{K} = \hat{C} = g.$

Hence, $\hat{K} = \alpha \hat{K} + (1-\alpha)\hat{H} \Rightarrow \boxed{g = \hat{K} = \hat{H} = \gamma u - \delta_H}$!!

- the greater is $u = \frac{H_H}{H}$, the faster is growth.
- in the optimal allocation, there will be $u \in (0,1)$ because one needs immediate output (and thus consumption) as well!

Growth with Externalities

- a model based on Romer (1986)
- capital accumulation increases total factor productivity (TFP)
- this effect is external to the firms — "learning by doing" externality

Key assumptions:

- production function as seen by firm $i \in [0,1]$:

$$Y_i = F(K_i, AL_i); \quad \text{with constant returns to scale (CRS)};$$

- upon aggregation,

$$\int_0^1 K_i di = K, \quad \int_0^1 L_i di = L \equiv \text{const},$$

- the externality takes the form:

$$A = BK, \quad B \equiv \text{const}.$$

$\boxed{!}$

Firms are perfectly competitive, so that

$$\tilde{r} = \frac{\partial Y_i}{\partial K_i}, \quad w = \frac{\partial Y_i}{\partial L_i}.$$

By symmetry, $\tilde{r} = \frac{\partial Y}{\partial K}, \quad w = \frac{\partial Y}{\partial L}$.

Using CRS and the definition of externality,

$$\frac{Y}{K} = \frac{F(K, BKL)}{K} = F(1, BL) := \tilde{f}(L)$$

$$\frac{Y}{L} = \frac{F(K, BKL)}{L} = F(k, BK) = \frac{Y}{K}\cdot\frac{K}{L} = k\cdot\tilde{f}(L).$$

Inserting we obtain
$$\tilde{r} = \tilde{f}(L) - L\tilde{f}'(L) \quad , \quad w = K\tilde{f}'(L).$$

<u>Note</u>:

- the wage rate is straightforward given $Y = K\tilde{f}(L)$
- $\tilde{r} \neq \frac{\partial(K\tilde{f}(L))}{\partial K} = \tilde{f}(L)$ because firms don't take the externality into account!
- instead, use the Euler theorem to obtain

$$(CRS) \quad Y = \frac{\partial Y}{\partial K}K + \frac{\partial Y}{\partial L}L \Rightarrow \frac{\partial Y}{\partial K} = \frac{1}{K}\left(Y - \frac{\partial Y}{\partial L}L\right) =$$
$$= \tilde{f}(L) - L\frac{\cancel{K}\tilde{f}'(L)}{\cancel{K}}.$$

<u>Households</u> maximize total discounted utility subject to the CRRA assumption and usual asset dynamics:
$$\max_c \int_0^\infty e^{-\rho t}u(c)dt, \quad \text{where } u(c) = \frac{c^{1-\theta}-1}{1-\theta},$$
$$\text{subject to} \quad \dot{a} = (r - \underbrace{n}_{n=0 \text{ Here}})a + w - c,$$

yielding $\boxed{\hat{c} = \frac{r-\rho}{\theta}}$. → Euler equation

In <u>equilibrium</u>, $a = k = \frac{K}{L}$, where $L \equiv$ fixed.

- Comparing with $\dot{K} = Y - C - \delta K$ we obtain $\underbrace{r}_{\text{NET RETURN}} = \underbrace{\tilde{r}}_{\text{GROSS RETURN}} - \delta$

and so
$$\boxed{\hat{y} = \hat{k} = \hat{c} = \frac{\tilde{f}(L) - L\tilde{f}'(L) - \delta - \rho}{\theta}}$$
→ <u>CONSTANT</u> growth rate of the economy $\boxed{!!}$

- The <u>transversality condition</u> requires that

$$\lim_{t\to\infty} \lambda a = 0.$$

It is sufficient that $\lim_{t\to\infty} \hat{\lambda} + \hat{a} < 0.$

For this to hold, we need

$$-r + \frac{r-\rho}{\theta} < 0 \iff (1-\theta) r < \rho \iff$$

$$\iff (1-\theta)\left(\tilde{f}(L) - L\tilde{f}'(L) - \delta\right) < \rho.$$

- Note that — just like in the case of the AK model — there is <u>no transitional dynamics</u>, $\hat{c} \equiv \text{const}$.
- Long-run growth driven by physical capital accumulation despite the fact that firms face decreasing returns to capital.
- The capital's share of output is

$$\pi_K = \frac{\partial Y}{\partial K} \cdot \frac{K}{Y} = \frac{\tilde{f}(L) - L\tilde{f}'(L)}{\tilde{f}(L)} = 1 - L\frac{\tilde{f}'(L)}{\tilde{f}(L)},$$

$$\pi_K \in (0, 1).$$

- <u>Scale effect</u>: $\hat{y} = \hat{k} = \hat{c}$ depends on L.